

On the Capacity of the Two-Hop Half-Duplex Relay Channel

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Abstract

Although extensively investigated, the capacity of the two-hop half-duplex (HD) relay channel is not fully understood. In particular, a capacity expression which can be easily evaluated is not available and an explicit coding scheme which achieves the capacity is not known either. In this paper, we derive a new expression for the capacity of the two-hop HD relay channel by simplifying previously derived converse expressions. Compared to previous results, the new capacity expression can be easily evaluated. Moreover, we propose an explicit coding scheme which achieves the capacity. To achieve the capacity, the relay does not only send information to the destination by transmitting information-carrying symbols but also with the zero symbols resulting from the relay's silence during reception. As examples, we compute the capacities of the two-hop HD relay channel for the cases when the source-relay and relay-destination links are both binary-symmetric channels (BSCs) and additive white Gaussian noise (AWGN) channels, respectively, and numerically compare the capacities with the rates achieved by conventional relaying where the relay receives and transmits in a codeword-by-codeword fashion and switches between reception and transmission in a strictly alternating manner. Our numerical results show that the capacities of the two-hop HD relay channel for BSC and AWGN links are significantly larger than the rates achieved with conventional relaying.

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I. INTRODUCTION

The relay channel is one of the building blocks of any general network. As such, it has been widely investigated in the literature e.g. [1]-[20]. The simplest relay channel is the two-hop relay channel comprised of a source, a relay, and a destination, where the direct link between the source and the destination is not available. In this relay channel, the source transmits a message to the relay, which then forwards it to the destination. Generally, a relay can employ two different modes of reception and transmission, i.e., full-duplex (FD) and half-duplex (HD). In the FD mode, the relay receives and transmits at the same time and in the same frequency band. In contrast, in the HD mode, the relay receives and transmits in the same frequency band but not at the same time or at the same time but in orthogonal frequency bands, in order to avoid self-interference. Given the limitations of current radio implementations, practical FD relaying suffers from self-interference. As a result, HD relaying has been widely adopted in the literature [2]-[5], [8], [9], [17]-[20].

The capacity of the two-hop FD relay channel without self-interference has been derived in [1] (see the capacity of the degraded relay channel). On the other hand, although extensively investigated, the capacity of the two-hop HD relay channel is not fully known nor understood. The reason for this is that a capacity expression which can be easily evaluated is not available and an explicit coding scheme which achieves the capacity is not known either. Currently, for HD relaying, detailed coding schemes exist only for rates which are strictly smaller than the capacity, see [2] and [3]. To achieve the rates given in [2] and [3], the HD relay receives a codeword in one time slot, decodes the received codeword, and re-encodes and re-transmits the decoded information in the following time slot. However, such fixed switching between reception and transmission at the HD relay was shown to be suboptimal in [4]. In particular, in [4], it was shown that if the fixed scheduling of reception and transmission at the HD relay is abandoned, then additional information can be encoded in the relay's reception and transmission switching pattern yielding an increase in data rate. In addition, it was shown in [4] that the HD relay channel can be analyzed using the framework developed for the FD relay channel in [1]. In particular, results derived for the FD relay channel in [1] can be directly applied to the HD relay channel. Thereby, using the converse for the degraded relay channel in [1], the capacity of the

discrete memoryless two-hop HD relay channel is obtained as [4], [5], [6]

$$C = \max_{p(x_1, x_2)} \min \{I(X_1; Y_1 | X_2), I(X_2; Y_2)\}, \quad (1)$$

where $I(\cdot; \cdot)$ denotes the mutual information, X_1 and X_2 are the inputs at source and relay, respectively, Y_1 and Y_2 are the outputs at relay and destination, respectively, and $p(x_1, x_2)$ is the joint probability mass function (PMF) of X_1 and X_2 . Moreover, it was shown in [4], [5], [6] that X_2 can be represented as $X_2 = [X'_2, U]$, where U is an auxiliary random variable with two outcomes t and r corresponding to the HD relay transmitting and receiving, respectively. Thereby, (1) can be written equivalently as

$$C = \max_{p(x_1, x'_2, u)} \min \{I(X_1; Y_1 | X'_2, U), I(X'_2, U; Y_2)\}, \quad (2)$$

where $p(x_1, x'_2, u)$ is the joint PMF of X_1 , X'_2 , and U . However, the capacity expressions in (1) and (2), respectively, cannot be evaluated since it is not known how X_1 and X_2 nor X_1 , X'_2 , and U are mutually dependent, i.e., $p(x_1, x_2)$ and $p(x_1, x'_2, u)$ are not known. In fact, the authors of [5, page 2552] state that: “*Despite knowing the capacity expression (i.e., expression (2)), its actual evaluation is elusive as it is not clear what the optimal input distribution $p(x_1, x'_2, u)$ is.*” On the other hand, for the coding scheme that would achieve (1) and (2) if $p(x_1, x_2)$ and $p(x_1, x'_2, u)$ were known, it can be argued that it has to be a decode-and-forward strategy since the two-hop HD relay channel belongs to the class of the degraded relay channels defined in [1]. Thereby, the HD relay should decode the received codewords, map the decoded information to new codewords, and transmit them to the destination. Moreover, it is known from [4] that such a coding scheme requires the HD relay to switch between reception and transmission in a symbol-by-symbol manner, and not in a codeword-by-codeword manner as in [2] and [3]. However, since $p(x_1, x_2)$ and $p(x_1, x'_2, u)$ are not known and since an explicit coding scheme does not exist, it is currently not known how to evaluate (1) and (2) nor how to encode additional information in the relay’s reception and transmission switching pattern and thereby achieve (1) and (2).

Motivated by the above discussion, in this paper, we derive a new expression for the capacity of the two-hop HD relay channel by simplifying previously derived converse expressions. In contrast to previous results, the new capacity expression can be easily evaluated. Moreover, we propose an explicit coding scheme which achieves the capacity. In particular, we show that

achieving the capacity requires the relay indeed to switch between reception and transmission in a symbol-by-symbol manner as predicted in [4]. Thereby, the relay does not only send information to the destination by transmitting information-carrying symbols but also with the zero symbols resulting from the relay's silence during reception. In addition, we propose a modified coding scheme for practical implementation where the HD relay receives and transmits at the same time (i.e., as in FD relaying), however, the simultaneous reception and transmission is performed such that self-interference is completely avoided. As examples, we compute the capacities of the two-hop HD relay channel for the cases when the source-relay and relay-destination links are both binary-symmetric channels (BSCs) and additive white Gaussian noise (AWGN) channels, respectively, and we numerically compare the capacities with the rates achieved by conventional relaying where the relay receives and transmits in a codeword-by-codeword fashion and switches between reception and transmission in a strictly alternating manner. Our numerical results show that the capacities of the two-hop HD relay channel for BSC and AWGN links are significantly larger than the rates achieved with conventional relaying.

The rest of this paper is organized as follows. In Section II, we present the channel model. In Section III, we derive a new expression for the capacity of the considered channel and propose a corresponding coding scheme. In Section IV, we investigate the capacity for the cases when the source-relay and relay-destination links are both BSCs and AWGN channels, respectively. In Section V, we numerically evaluate the derived capacity expressions and compare them to the rates achieved by conventional relaying. Finally, Section VI concludes the paper.

II. SYSTEM MODEL

The two-hop HD relay channel consists of a source, an HD relay, and a destination, and the direct link between source and destination is not available, see Fig. 1. Due to the HD constraint, the relay cannot transmit and receive at the same time. In the following, we formally define the channel model.

A. Channel Model

The discrete memoryless two-hop HD relay channel is defined by \mathcal{X}_1 , \mathcal{X}_2 , \mathcal{Y}_1 , \mathcal{Y}_2 , and $p(y_1, y_2|x_1, x_2)$, where \mathcal{X}_1 and \mathcal{X}_2 are the finite input alphabets at the source and the relay, respectively, \mathcal{Y}_1 and \mathcal{Y}_2 are the finite output alphabets at the relay and the destination, respectively, and



Fig. 1. Two-hop relay channel.

$p(y_1, y_2 | x_1, x_2)$ is the PMF on $\mathcal{Y}_1 \times \mathcal{Y}_2$ for given $x_1 \in \mathcal{X}_1$ and $x_2 \in \mathcal{X}_2$. The channel is memoryless in the sense that given the input symbols for the i -th channel use, the i -th output symbols are independent from all previous input symbols. As a result, the conditional PMF $p(y_1^n, y_2^n | x_1^n, x_2^n)$, where the notation a^n is used to denote the ordered sequence $a^n = (a_1, a_2, \dots, a_n)$, can be factorized as $p(y_1^n, y_2^n | x_1^n, x_2^n) = \prod_{i=1}^n p(y_{1i}, y_{2i} | x_{1i}, x_{2i})$.

For the considered channel and the i -th channel use, let X_{1i} and X_{2i} denote the random variables (RVs) which model the input at source and relay, respectively, and let Y_{1i} and Y_{2i} denote the RVs which model the output at relay and destination, respectively.

In the following, we model the HD constraint of the relay and discuss its effect on some important PMFs that will be used throughout this paper.

B. Mathematical Modelling of the HD Constraint

Due to the HD constraint of the relay, the input and output symbols of the relay cannot assume non-zero values at the same time. More precisely, for each channel use, if the input symbol of the relay is non-zero then the output symbol has to be zero, and vice versa, if the output symbol of the relay is non-zero then the input symbol has to be zero. Hence, the following holds

$$Y_{1i} = \begin{cases} Y'_{1i} & \text{if } X_{2i} = 0 \\ 0 & \text{if } X_{2i} \neq 0, \end{cases} \quad (3)$$

where Y'_{1i} is an RV that take values from set \mathcal{Y}_1 .

In order to model the HD constraint of the relay more conveniently, we represent the input set of the relay \mathcal{X}_2 as the union of two sets $\mathcal{X}_2 = \mathcal{X}_{2R} \cup \mathcal{X}_{2T}$, where \mathcal{X}_{2R} contains only one element, the zero symbol, and \mathcal{X}_{2T} contains all symbols in \mathcal{X}_2 except the zero symbol. Note that, because of the HD constraint, \mathcal{X}_2 has to contain the zero symbol, i.e., the relay has to be silent

in some portion of the time during which the relay can receive. Furthermore, we introduce an auxiliary random variable, denoted by U_i , which takes values from the set $\{t, r\}$, where t and r correspond to the relay transmitting a non-zero symbol and a zero symbol, respectively. Hence, U_i is defined as

$$U_i = \begin{cases} r & \text{if } X_{2i} = 0 \\ t & \text{if } X_{2i} \neq 0. \end{cases} \quad (4)$$

Let us denote the probabilities of the relay transmitting a non-zero and a zero symbol for the i -th channel use as $\Pr\{U_i = t\} = \Pr\{X_{2i} \neq 0\} = P_{U_i}$ and $\Pr\{U_i = r\} = \Pr\{X_{2i} = 0\} = 1 - P_{U_i}$, respectively. We now use (4) and represent X_{2i} as a function of the outcome of U_i . Hence, we have

$$X_{2i} = \begin{cases} 0 & \text{if } U_i = r \\ V_i & \text{if } U_i = t, \end{cases} \quad (5)$$

where V_i is an RV with distribution $p_{V_i}(x_{2i})$ that takes values from the set \mathcal{X}_{2T} , or equivalently, an RV which takes values from the set \mathcal{X}_2 , but with $p_{V_i}(x_{2i} = 0) = 0$. From (5), we obtain

$$p(x_{2i}|U_i = r) = \delta(x_{2i}), \quad (6)$$

$$p(x_{2i}|U_i = t) = p_{V_i}(x_{2i}), \quad (7)$$

where $\delta(x) = 1$ if $x = 0$ and $\delta(x) = 0$ if $x \neq 0$. Furthermore, for the derivation of the capacity, we will also need the conditional PMF $p(x_{1i}|x_{2i} = 0)$ which is the input distribution at the source when the relay transmits a zero (i.e., when $U_i = r$). As we will see in Theorem 1, the distributions $p(x_{1i}|x_{2i} = 0)$ and $p_{V_i}(x_{2i})$ have to be optimized in order to achieve the capacity. Using $p(x_{2i}|U_i = r)$ and $p(x_{2i}|U_i = t)$, and the law of total probability, the PMF of X_{2i} , $p(x_{2i})$, is obtained as

$$\begin{aligned} p(x_{2i}) &= p(x_{2i}|U_i = t)P_{U_i} + p(x_{2i}|U_i = r)(1 - P_{U_i}) \\ &\stackrel{(a)}{=} p_{V_i}(x_{2i})P_{U_i} + \delta(x_{2i})(1 - P_{U_i}), \end{aligned} \quad (8)$$

where (a) follows from (6) and (7). In addition, we will also need the distribution of Y_{2i} , $p(y_{2i})$, which, using the law of total probability, can be written as

$$p(y_{2i}) = p(y_{2i}|U_i = t)P_{U_i} + p(y_{2i}|U_i = r)(1 - P_{U_i}). \quad (9)$$

On the other hand, using X_{2i} and the law of total probability, $p(y_{2i}|U_i = r)$ can be written as

$$\begin{aligned}
p(y_{2i}|U_i = r) &= \sum_{x_{2i} \in \mathcal{X}_2} p(y_{2i}, x_{2i}|U_i = r) = \sum_{x_{2i} \in \mathcal{X}_2} p(y_{2i}|x_{2i}, U_i = r)p(x_{2i}|U_i = r) \\
&\stackrel{(a)}{=} \sum_{x_{2i} \in \mathcal{X}_2} p(y_{2i}|x_{2i}, U_i = r)\delta(x_{2i}) = p(y_{2i}|x_{2i} = 0, U_i = r) \\
&\stackrel{(b)}{=} p(y_{2i}|x_{2i} = 0),
\end{aligned} \tag{10}$$

where (a) is due to (6) and (b) is the result of conditioning on the same variable twice since if $X_{2i} = 0$ then $U_i = r$, and vice versa. On the other hand, using X_{2i} and the law of total probability, $p(y_{2i}|U_i = t)$ can be written as

$$\begin{aligned}
p(y_{2i}|U_i = t) &= \sum_{x_{2i} \in \mathcal{X}_2} p(y_{2i}, x_{2i}|U_i = t) = \sum_{x_{2i} \in \mathcal{X}_2} p(y_{2i}|x_{2i}, U_i = t)p(x_{2i}|U_i = t) \\
&\stackrel{(a)}{=} \sum_{x_{2i} \in \mathcal{X}_{2T}} p(y_{2i}|x_{2i}, U_i = t)p_{V_i}(x_{2i}) \stackrel{(b)}{=} \sum_{x_{2i} \in \mathcal{X}_{2T}} p(y_{2i}|x_{2i})p_{V_i}(x_{2i}),
\end{aligned} \tag{11}$$

where (a) follows from (7) and since V_i takes values from set \mathcal{X}_{2T} , and (b) follows since conditioned on X_{2i} , Y_{2i} is independent of U_i . In (11), $p(y_{2i}|x_{2i})$ is the distribution at the output of the relay-destination channel conditioned on the relay's input X_{2i} .

C. Mutual Information and Entropy

For the capacity expression given later in Theorem 1, we need $I(X_1; Y_1|X_2 = 0)$, which is the mutual information between the source's input X_1 and the relay's output Y_1 conditioned on the relay having its input set to $X_2 = 0$, and $I(X_2; Y_2)$, which is the mutual information between the relay's input X_2 and the destination's output Y_2 .

The mutual information $I(X_1; Y_1|X_2 = 0)$ is obtained by definition as

$$I(X_1; Y_1|X_2 = 0) = \sum_{x_1 \in \mathcal{X}_1} \sum_{y_1 \in \mathcal{Y}_1} p(y_1|x_1, x_2 = 0)p(x_1|x_2 = 0) \log_2 \left(\frac{p(y_1|x_1, x_2 = 0)}{p(y_1|x_2 = 0)} \right), \tag{12}$$

where

$$p(y_1|x_2 = 0) = \sum_{x_1 \in \mathcal{X}_1} p(y_1|x_1, x_2 = 0)p(x_1|x_2 = 0). \tag{13}$$

In (12) and (13), $p(y_1|x_1, x_2 = 0)$ is the distribution at the output of the source-relay channel conditioned on the relay having its input set to $X_2 = 0$, and conditioned on the input symbols at the source X_1 .

On the other hand, $I(X_2; Y_2)$ is given by

$$I(X_2; Y_2) = H(Y_2) - H(Y_2|X_2), \quad (14)$$

where $H(Y_2)$ is the entropy of RV Y_2 , and $H(Y_2|X_2)$ is the entropy of Y_2 conditioned on X_2 .

The entropy $H(Y_2)$ can be found by definition as

$$\begin{aligned} H(Y_2) &= - \sum_{y_2 \in \mathcal{Y}_2} p(y_2) \log_2(p(y_2)) \\ &\stackrel{(a)}{=} - \sum_{y_2 \in \mathcal{Y}_2} [p(y_2|U=t)P_U + p(y_2|U=r)(1-P_U)] \\ &\quad \times \log_2 [p(y_2|U=t)P_U + p(y_2|U=r)(1-P_U)], \end{aligned} \quad (15)$$

where (a) follows from (9). Now, inserting $p(y_2|U=r)$ and $p(y_2|U=t)$ given in (10) and (11), respectively, into (15), we obtain the final expression for $H(Y_2)$, as

$$\begin{aligned} H(Y_2) &= - \sum_{y_2 \in \mathcal{Y}_2} \left[P_U \sum_{x_2 \in \mathcal{X}_{2T}} p(y_2|x_2)p_V(x_2) + p(y_2|x_2=0)(1-P_U) \right] \\ &\quad \times \log_2 \left[P_U \sum_{x_2 \in \mathcal{X}_{2T}} p(y_2|x_2)p_V(x_2) + p(y_2|x_2=0)(1-P_U) \right]. \end{aligned} \quad (16)$$

On the other hand, the conditional entropy $H(Y_2|X_2)$ can be found based on its definition as

$$\begin{aligned} H(Y_2|X_2) &= - \sum_{x_2 \in \mathcal{X}_2} p(x_2) \sum_{y_2 \in \mathcal{Y}_2} p(y_2|x_2) \log_2(p(y_2|x_2)) \\ &\stackrel{(a)}{=} - P_U \sum_{x_2 \in \mathcal{X}_{2T}} p_V(x_2) \sum_{y_2 \in \mathcal{Y}_2} p(y_2|x_2) \log_2(p(y_2|x_2)) \\ &\quad - (1-P_U) \sum_{y_2 \in \mathcal{Y}_2} p(y_2|x_2=0) \log_2(p(y_2|x_2=0)), \end{aligned} \quad (17)$$

where (a) follows by inserting $p(x_2)$ given in (8). Inserting $H(Y_2)$ and $H(Y_2|X_2)$ given in (16) and (17), respectively, into (14), we obtain the final expression for $I(X_2; Y_2)$, which is dependent on $p(x_2)$, i.e., on $p_V(x_2)$ and P_U . To highlight this dependence, we sometimes write $I(X_2; Y_2)$ as $I(X_2; Y_2) \Big|_{P_U}$.

We are now ready to present a new capacity expression for the considered relay channel.

III. NEW CAPACITY EXPRESSION AND EXPLICIT CODING SCHEME

In this section, we provide a new and easy-to-evaluate expression for the capacity of the two-hop HD relay channel by simplifying previously derived converse expressions and provide an explicit coding scheme which achieves the capacity.

A. New Capacity Expression

A new expression for the capacity of the two-hop HD relay channel is given in the following theorem.

Theorem 1: The capacity of the two-hop HD relay channel is given by

$$C = \max_{P_U} \min \left\{ \max_{p(x_1|x_2=0)} I(X_1; Y_1|X_2=0)(1-P_U), \max_{p_V(x_2)} I(X_2; Y_2)|_{P_U} \right\}, \quad (18)$$

where $I(X_1; Y_1|X_2=0)$ is given in (12) and $I(X_2; Y_2)$ is given in (14)-(17). The optimal P_U that maximizes the capacity in (18) is given by $P_U^* = \min\{P'_U, P''_U\}$, where P'_U is the solution of

$$\max_{p(x_1|x_2=0)} I(X_1; Y_1|X_2=0)(1-P_U) = \max_{p_V(x_2)} I(X_2; Y_2)|_{P_U}, \quad (19)$$

where, if (19) has two solutions, then P'_U is the smaller of the two, and P''_U is the solution of

$$\frac{\partial \left(\max_{p_V(x_2)} I(X_2; Y_2)|_{P_U} \right)}{\partial P_U} = 0. \quad (20)$$

If $P_U^* = P'_U$, the capacity in (18) simplifies to

$$C = \max_{p(x_1|x_2=0)} I(X_1; Y_1|X_2=0)(1-P'_U) = \max_{p_V(x_2)} I(X_2; Y_2)|_{P_U=P'_U}, \quad (21)$$

whereas, if $P_U^* = P''_U$, the capacity in (18) simplifies to

$$C = \max_{p_V(x_2)} I(X_2; Y_2)|_{P_U=P''_U} = \max_{p(x_2)} I(X_2; Y_2), \quad (22)$$

which is the capacity of the relay-destination channel.

Proof: To derive the new capacity expression in (18), we combine the results from [1] and [4]. In particular, [4] showed that the HD relay channel can be analyzed with the framework developed for the FD relay channel in [1]. Since the considered two-hop HD relay channel belongs to the class of degraded relay channels defined in [1], the rate of this channel is upper bounded by [1], [4]

$$R \leq \max_{p(x_1, x_2)} \min \{I(X_1; Y_1|X_2), I(X_2; Y_2)\}. \quad (23)$$

On the other hand, $I(X_1; Y_1|X_2)$ can be simplified as

$$\begin{aligned} I(X_1; Y_1|X_2) &= I(X_1; Y_1|X_2=0)(1-P_U) + I(X_1; Y_1|X_2 \neq 0)P_U \\ &\stackrel{(a)}{=} I(X_1; Y_1|X_2=0)(1-P_U), \end{aligned} \quad (24)$$

where (a) follows from (3) since when $X_2 \neq 0$, Y_1 is deterministically zero thereby leading to $I(X_1; Y_1 | X_2 \neq 0) = 0$. Inserting (24) into (23), we obtain

$$R \leq \max_{p(x_1, x_2)} \min \{ I(X_1; Y_1 | X_2 = 0)(1 - P_U), I(X_2; Y_2) \}. \quad (25)$$

Note that the PMF $p(x_1, x_2)$ can be written equivalently as

$$p(x_1, x_2) = p(x_1 | x_2)p(x_2) \stackrel{(a)}{=} p(x_1 | x_2)(p_V(x_2)P_U + \delta(x_2)(1 - P_U)), \quad (26)$$

where (a) follows from (8). As a result, the maximization over $p(x_1, x_2)$ can be resolved into a joint maximization over $p(x_1 | x_2)$, $p_V(x_2)$, and P_U . Thereby, (25) can be written equivalently as

$$R \leq \max_{p(x_1 | x_2), p_V(x_2), P_U} \min \{ I(X_1; Y_1 | X_2 = 0)(1 - P_U), I(X_2; Y_2) \}. \quad (27)$$

Now, note that $I(X_1; Y_1 | X_2 = 0)$ and $I(X_2; Y_2)$ are dependent only on $p(x_1 | x_2 = 0)$ and $p_V(x_2)$, respectively, and no other function in the right hand side of (27) is dependent on $p(x_1 | x_2 = 0)$ and $p_V(x_2)$. Therefore, (27) can be written equivalently as

$$R \leq \max_{P_U} \min \left\{ \max_{p(x_1 | x_2 = 0)} I(X_1; Y_1 | X_2 = 0)(1 - P_U), \max_{p_V(x_2)} I(X_2; Y_2) \Big|_{P_U} \right\}, \quad (28)$$

where the mutual informations $I(X_1; Y_1 | X_2 = 0)$ and $I(X_2; Y_2) \Big|_{P_U}$ are concave functions with respect to $p(x_1 | x_2 = 0)$ and $p_V(x_2)$, respectively¹. Now, note that the right hand side of the expression in (28) is identical to the capacity expression in (18). The rest of the theorem follows from solving (18) with respect to P_U , and simplifying the result. In particular, note that the first term inside the $\min\{\cdot\}$ function in (18) is a decreasing function with respect to P_U . This function achieves its maximum for $P_U = 0$ and its minimum, which is zero, for $P_U = 1$. On the other hand, the second term inside the $\min\{\cdot\}$ function in (18) is a concave function with respect to P_U ². Moreover, this function is zero when $P_U = 0$, i.e., when the relay is always silent. Now, the maximization of the minimum of the decreasing and concave functions with respect to P_U , given in (18), has a solution $P_U = P_U''$, when the concave function reaches its

¹For concavity of $I(X_1; Y_1 | X_2 = 0)$ and $I(X_2; Y_2) \Big|_{P_U}$ with respect to $p(x_1 | x_2 = 0)$ and $p(x_2)$, respectively, see [21]. On the other hand, since $1 - P_U$ is just the probability $p(x_2 = 0)$ and since $p_V(x_2)$ contains the rest of the probability constrained parameters in $p(x_2)$, $I(X_2; Y_2)$ is a jointly concave function with respect to $p_V(x_2)$ and P_U .

²In [22, pp. 87-88], it is proven that if $f(x, y)$ is a jointly concave function in both (x, y) and \mathcal{C} is a convex nonempty set, then the function $g(x) = \max_{y \in \mathcal{C}} f(x, y)$ is concave in x . Using this result, and noting that $I(X_2; Y_2)$ is a jointly concave function with respect to $p_V(x_2)$ and P_U , we can conclude that $\max_{p_V(x_2)} I(X_2; Y_2) \Big|_{P_U}$ is concave with respect to P_U .

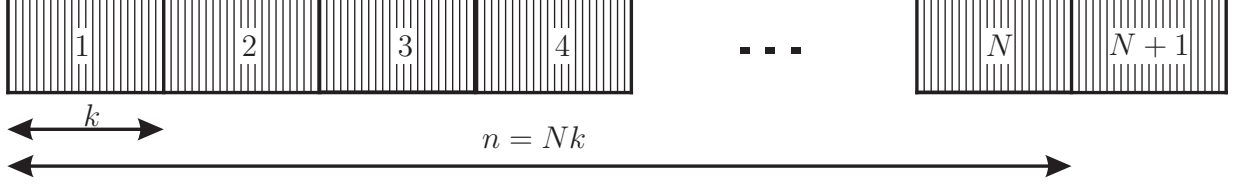


Fig. 2. To achieve the capacity in (18), transmission is organized in $N + 1$ blocks and each block comprises k channel uses.

maximum, found from (20), and when for this point, i.e., for $P_U = P_U''$, the decreasing function is larger than the concave function. Otherwise, the solution is $P_U = P_U'$ which is found from (19) and in which case $P_U' < P_U''$ holds. If (19) has two solutions, then P_U' has to be the smaller of the two since $\max_{p(x_1|x_2=0)} I(X_1; Y_1 | X_2 = 0)(1 - P_U)$ is a decreasing function with respect to P_U . Now, when $P_U^* = P_U'$, (19) holds, (18) simplifies to (21). Whereas, when $P_U^* = P_U''$, then $\max_{p_V(x_2)} I(X_2; Y_2) \big|_{P_U = P_U''} = \max_{p_V(x_2)} \max_{P_U} I(X_2; Y_2) = \max_{p(x_2)} I(X_2; Y_2)$, thereby leading to (22). This concludes the proof. \blacksquare

B. Explicit Capacity Achieving Coding Scheme

Since an explicit capacity coding scheme is not available in the literature, in the following, we propose an explicit coding scheme which achieves the capacity in (18).

In the following, we describe a method for transferring nR bits of information in $n + k$ channel uses, where $n, k \rightarrow \infty$ and $n/(n + k) \rightarrow 1$ as $n, k \rightarrow \infty$. As a result, the information is transferred at rate R . To this end, the transmission is carried out in $N + 1$ blocks, where $N \rightarrow \infty$. In each block, we use the channel k times. The numbers N and k are chosen such that $n = Nk$ holds. The transmission in $N + 1$ blocks is illustrated in Fig. 2.

The source transmits message W , drawn uniformly from message set $\{1, 2, \dots, 2^{nR}\}$, from the source via the HD relay to the destination. To this end, before the start of transmission, message W is spilt into N messages, denoted by $w(1), \dots, w(N)$, where each $w(i)$, $\forall i$, contains kR bits of information. The transmission is carried out in the following manner. In block one, the source sends message $w(1)$ in k channel uses to the relay and the relay is silent. In block i , for $i = 2, \dots, N$, source and relay send messages $w(i)$ and $w(i - 1)$ to relay and destination, respectively, in k channel uses. In block $N + 1$, the relay sends message $w(N)$ in k channel uses to the destination and the source is silent. Hence, in the first block and in the $(N + 1)$ -th

block, the relay and the source are silent, respectively, since in the first block the relay does not have information to transmit, and in block $N + 1$, the source has no more information to transmit. In blocks 2 to N , both source and relay transmit, while meeting the HD constraint in every channel use. Hence, during the $N + 1$ blocks, the channel is used $k(N + 1)$ times to send $nR = NkR$ bits of information, leading to an overall information rate given by

$$\lim_{N \rightarrow \infty} \lim_{k \rightarrow \infty} \frac{NkR}{k(N + 1)} = R \text{ bits/use.} \quad (29)$$

A detailed description of the proposed coding scheme is given in the following, where we explain the rates, codebooks, encoding, and decoding used for transmission.

Rates: The transmission rate of both source and relay is denoted by R and given by

$$R = C - \epsilon, \quad (30)$$

where C is given in Theorem 1 and $\epsilon > 0$ is an arbitrarily small number. Note that R is a function of P_U^* , see Theorem 1.

Codebooks: We have two codebooks: The source's transmission codebook and the relay's transmission codebook.

The source's transmission codebook is generated by mapping each possible binary sequence comprised of kR bits, where R is given by (30), to a codeword³ $\mathbf{x}_{1|r}$ comprised of $k(1 - P_U^*)$ symbols. The symbols in each codeword $\mathbf{x}_{1|r}$ are generated independently according to distribution $p(x_1|x_2 = 0)$. Since in total there are 2^{kR} possible binary sequences comprised of kR bits, with this mapping we generate 2^{kR} codewords $\mathbf{x}_{1|r}$ each containing $k(1 - P_U^*)$ symbols. These 2^{kR} codewords form the source's transmission codebook, which we denote by $\mathcal{C}_{1|r}$.

The relay's transmission codebook is generated by mapping each possible binary sequence comprised of kR bits, where R is given by (30), to a transmission codeword \mathbf{x}_2 comprised of k symbols. The i -th symbol, $i = 1, \dots, k$, in codeword \mathbf{x}_2 is generated in the following manner. For each symbol a coin is tossed. The coin is such that it produces symbol r with probability $1 - P_U^*$ and symbol t with probability P_U^* . If the outcome of the coin flip is r , then the i -th symbol of the relay's transmission codeword \mathbf{x}_2 is set to zero. Otherwise, if the outcome of the coin flip is t , then the i -th symbol of codeword \mathbf{x}_2 is generated independently according to distribution

³The subscript $1|r$ in $\mathbf{x}_{1|r}$ is used to indicate that codeword $\mathbf{x}_{1|r}$ is comprised of symbols which are transmitted by the source only when $U_i = r$, i.e., when $X_{2i} = 0$.

$p_V(x_2)$. In this way, the symbols in \mathbf{x}_2 are distributed according to the distribution $p(x_2)$ given in (8). The 2^{kR} codewords \mathbf{x}_2 form the relay's transmission codebook denoted by \mathcal{C}_2 .

The two codebooks are known at all three nodes.

Encoding, Transmission, and Decoding: In the first block, the source maps $w(1)$ to the appropriate codeword $\mathbf{x}_{1|r}(1)$ from its codebook $\mathcal{C}_{1|r}$. Then, codeword $\mathbf{x}_{1|r}(1)$ is transmitted to the relay, which is scheduled to always receive and be silent (i.e., to set its input to zero) during the first block. However, knowing that the transmitted codeword from the source $\mathbf{x}_{1|r}(1)$ is comprised of $k(1 - P_U^*)$ symbols, the relay constructs the received codeword, denoted by $\mathbf{y}_{1|r}(1)$, only from the first $k(1 - P_U^*)$ received symbols. In Appendix A, we prove that codeword $\mathbf{x}_{1|r}(1)$ sent in the first block can be decoded successfully from the received codeword at the relay $\mathbf{y}_{1|r}(1)$ using a typical decoder [21] since R satisfies

$$R < \max_{p(x_1|x_2=0)} I(X_1; Y_1 | X_2 = 0)(1 - P_U^*). \quad (31)$$

In blocks $i = 2, \dots, N$, the encoding, transmission, and decoding are performed as follows. In blocks $i = 2, \dots, N$, the source and the relay map $w(i)$ and $w(i - 1)$ to the appropriate codewords $\mathbf{x}_{1|r}(i)$ and $\mathbf{x}_2(i)$ from codebooks $\mathcal{C}_{1|r}$ and \mathcal{C}_2 , respectively. Note that the source also knows $\mathbf{x}_2(i)$ since $\mathbf{x}_2(i)$ was generated from $w(i - 1)$ which the source transmitted in the previous (i.e., $(i - 1)$ -th) block. The transmission of $\mathbf{x}_{1|r}(i)$ and $\mathbf{x}_2(i)$ can be performed in two ways: 1) by the relay switching between reception and transmission, and 2) by the relay always receiving and transmitting as in FD relaying. We first explain the first option.

Note that both source and relay know the position of the zero symbols in $\mathbf{x}_2(i)$. Hence, if the first symbol in codeword $\mathbf{x}_2(i)$ is zero, then in the first symbol interval of block i , the source transmits its first symbol from codeword $\mathbf{x}_{1|r}(i)$ and the relay receives. By receiving, the relay actually also sends the first symbol of codeword $\mathbf{x}_2(i)$, which is the symbol zero, i.e., $x_{21} = 0$. On the other hand, if the first symbol in codeword $\mathbf{x}_2(i)$ is non-zero, then in the first symbol interval of block i , the relay transmits its first symbol from codeword $\mathbf{x}_2(i)$ and the source is silent. The same procedure is performed for the j -th channel use in block i , for $j = 1, \dots, k$. In particular, if the j -th symbol in codeword $\mathbf{x}_2(i)$ is zero, then in the j -th channel use of block i the source transmits its next untransmitted symbol from codeword $\mathbf{x}_{1|r}(i)$ and the relay receives. With this reception, the relay actually also sends the j -th symbol of codeword $\mathbf{x}_2(i)$, which is the symbol zero, i.e., $x_{2j} = 0$. On the other hand, if the j -th symbol in codeword

$\mathbf{x}_2(i)$ is non-zero, then for the j -th channel use of block i , the relay transmits the j -th symbol of codeword $\mathbf{x}_2(i)$ and the source is silent. Note that codeword $\mathbf{x}_2(i)$ contains $k(1 - P_U^*) \pm \varepsilon(i)$ symbols zeros, where $\varepsilon(i) > 0$. Due to the strong law of large numbers [21], $\lim_{k \rightarrow \infty} \varepsilon(i)/k = 0$ holds, which means that for large enough k , the fraction of symbols zeros in codeword $\mathbf{x}_2(i)$ is $1 - P_U^*$. Hence, for $k \rightarrow \infty$, the source can transmit practically all⁴ of its $k(1 - P_U^*)$ symbols from codeword $\mathbf{x}_{1|r}(i)$ during a single block to the relay. Let $\mathbf{y}_{1|r}(i)$ denote the corresponding received codeword at the relay. In Appendix A, we prove that the codewords $\mathbf{x}_{1|r}(i)$ sent in blocks $i = 2, \dots, N$ can be decoded successfully at the relay from the corresponding received codewords $\mathbf{y}_{1|r}(i)$ using a typical decoder [21] since R satisfies (31). Moreover, in Appendix A, we also prove that, for $k \rightarrow \infty$, the codewords $\mathbf{x}_{1|r}(i)$ can be successfully decoded at the relay even though, for some blocks $i = 2, \dots, N$, only $k(1 - P_U^*) - \varepsilon(i)$ symbols out of $k(1 - P_U^*)$ symbols in codewords $\mathbf{x}_{1|r}(i)$ are transmitted to the relay. On the other hand, the relay sends the entire codeword $\mathbf{x}_2(i)$, comprised of k symbols of which a fraction $1 - P_U^*$ are zeros, to the destination. In particular, the relay sends the zero symbols of codeword $\mathbf{x}_2(i)$ to the destination by being silent during reception, and sends the non-zero symbols of codeword $\mathbf{x}_2(i)$ to the destination by actually transmitting them. On the other hand, the destination listens during the entire block and receives a codeword $\mathbf{y}_2(i)$. By following the “standard” method in [21, Sec. 7.7] for analyzing the probability of error for rates smaller than the capacity, it can be shown in a straightforward manner that the destination can successfully decode $\mathbf{x}_2(i)$ from the received codeword $\mathbf{y}_2(i)$, and thereby obtain $w(i - 1)$, since rate R satisfies

$$R < \max_{p_V(x_2)} I(X_2; Y_2) \Big|_{P_U = P_U^*}. \quad (32)$$

In a practical implementation, the relay may not be able to switch between reception and transmission in a symbol-by-symbol manner, due to practical constraints regarding the speed of switching. Instead, we may allow the relay to receive and transmit at the same time and in the same frequency band similar to FD relaying. However, this simultaneous reception and transmission is performed while avoiding self-interference since, in each symbol interval, either the input or the output information-carrying symbol of the relay is zero. This is accomplished in the following manner. The source performs the same operations as for the case when the

⁴When we say practically all, we mean either all or all except for a negligible fraction $\lim_{k \rightarrow \infty} \varepsilon(i)/k = 0$ of them.

$\mathbf{x}_{1 r}:$	$x_{11 r}$	$x_{12 r}$	$x_{13 r}$	$x_{14 r}$				
$\mathbf{x}_1:$	$x_{11 r}$	0	0	$x_{12 r}$	0	$x_{13 r}$	$x_{14 r}$	0
$\mathbf{y}_1:$	y_{11}	y_{12}	y_{13}	y_{14}	y_{15}	y_{16}	y_{17}	y_{18}
$\mathbf{y}_{1 r}:$	y_{11}	y_{14}	y_{16}	y_{17}				
$\mathbf{x}_2:$	0	x_{22}	x_{23}	0	x_{25}	0	0	x_{28}
$\mathbf{y}_2:$	y_{21}	y_{22}	y_{23}	y_{24}	y_{25}	y_{26}	y_{27}	y_{28}

Fig. 3. Example of generated switching vector along with input/output codewords at source, relay, and destination.

relay switches between reception and transmission. On the other hand, the relay transmits all symbols from $\mathbf{x}_2(i)$ while continuously listening. Then, the relay discards from the received codeword, denoted by $\mathbf{y}_1(i)$, those symbols for which the corresponding symbols in $\mathbf{x}_2(i)$ are non-zero, and only collects the symbols in $\mathbf{y}_1(i)$ for which the corresponding symbols in $\mathbf{x}_2(i)$ are equal to zero. The collected symbols from $\mathbf{y}_1(i)$ constitute the relay's information-carrying received codeword $\mathbf{y}_{1|r}(i)$ which is used for decoding. Codeword $\mathbf{y}_{1|r}(i)$ is completely free of self-interference since the symbols in $\mathbf{y}_{1|r}(i)$ were received in symbol intervals for which the corresponding transmit symbol at the relay was zero.

In the last (i.e., the $(N + 1)$ -th) block, the source is silent and the relay transmits $w(N)$ by mapping it to the corresponding codeword $\mathbf{x}_2(i)$ from set \mathcal{C}_2 . The relay transmits all symbols in codeword $\mathbf{x}_2(i)$ to the destination. The destination can decode the received codeword in block $N + 1$ successfully, since (32) holds.

Finally, since both relay and destination can decode their respective codewords in each block, the entire message W can be decoded successfully at the destination at the end of the $(N + 1)$ -th block.

1) Coding Example: In Fig. 3, we show an example for vectors $\mathbf{x}_{1|r}$, \mathbf{x}_1 , \mathbf{y}_1 , $\mathbf{y}_{1|r}$, \mathbf{x}_2 , and \mathbf{y}_2 , for $k = 8$ and $P_U^* = 1/2$, where \mathbf{x}_1 contains all k input symbols at the source including the silences. From this example, it can be seen that \mathbf{x}_1 contains zeros due to silences for channel

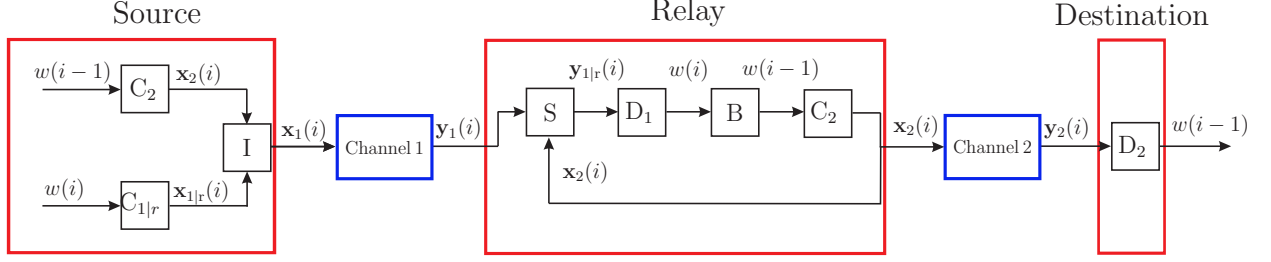


Fig. 4. Block diagram of the proposed channel coding protocol for time slot i . The following notations are used in the block diagram: $C_{1|r}$ and C_2 are encoders, D_1 and D_2 are decoders, I is an inserter, S is a selector, B is a buffer, and $w(i)$ denotes the message transmitted by the source in block i .

uses for which the corresponding symbol in \mathbf{x}_2 is non-zero. By comparing \mathbf{x}_1 and \mathbf{x}_2 it can be seen that the HD constraint is satisfied for each symbol duration.

The block diagram of the proposed coding scheme is shown in Fig. 4. In particular, in Fig 4, we show schematically the encoding, transmission, and decoding at source, relay, and destination. The flow of encoding/decoding in Fig. 4 is as follows. Messages $w(i-1)$ and $w(i)$ are encoded into $\mathbf{x}_2(i)$ and $\mathbf{x}_{1|r}(i)$, respectively, at the source using the encoders C_2 and $C_{1|r}$, respectively. Then, an inserter I is used to create the vector $\mathbf{x}_1(i)$ by inserting the symbols of $\mathbf{x}_{1|r}(i)$ into the positions of $\mathbf{x}_1(i)$ for which the corresponding elements of $\mathbf{x}_2(i)$ are zeros and setting all other symbols in $\mathbf{x}_1(i)$ to zero. The source then transmits $\mathbf{x}_1(i)$. On the other hand, the relay, encodes $w(i-1)$ into $\mathbf{x}_2(i)$ using encoder C_2 . Then, the relay transmits $\mathbf{x}_2(i)$ while receiving $\mathbf{y}_1(i)$. Next, using $\mathbf{x}_2(i)$, the relay constructs $\mathbf{y}_{1|r}(i)$ from $\mathbf{y}_1(i)$ by selecting only those symbols for which the corresponding symbol in $\mathbf{x}_2(i)$ is zero. The relay then decodes $\mathbf{y}_{1|r}(i)$, using decoder D_1 , into $w(i)$ and stores the decoded bits in its buffer B . The destination receives $\mathbf{y}_2(i)$, and decodes it using decoder D_2 , into $w(i-1)$.

IV. CAPACITY EXAMPLES

In the following, we evaluate the capacity of the considered relay channel when the source-relay and relay-destination links are both BSCs and AWGN channels, respectively.

A. Binary Symmetric Channels

Assume that the source-relay and relay-destination links are both BSCs, where $\mathcal{X}_1 = \mathcal{X}_2 = \mathcal{Y}_1 = \mathcal{Y}_2 = \{0, 1\}$, with probability of error $P_{\varepsilon 1}$ and $P_{\varepsilon 2}$, respectively. Now, in order to obtain the

capacity for this relay channel, according to Theorem 1, we first have to find $\max_{p(x_1|x_2=0)} I(X_1; Y_1|X_2 = 0)$ and $\max_{p_V(x_2)} I(X_2; Y_2)$. For the BSC, the expression for $\max_{p(x_1|x_2=0)} I(X_1; Y_1|X_2 = 0)$ is well known and given by [1]

$$\max_{p(x_1|x_2=0)} I(X_1; Y_1|X_2 = 0) = 1 - H(P_{\varepsilon 1}), \quad (33)$$

where $H(P_{\varepsilon 1})$ is the binary entropy function, which for probability P is defined as

$$H(P) = -P \log_2(P) - (1 - P) \log_2(1 - P). \quad (34)$$

The distribution that maximizes $I(X_1; Y_1|X_2 = 0)$ is also well known and given by [1]

$$p(x_1 = 0|x_2 = 0) = p(x_1 = 1|x_2 = 0) = \frac{1}{2}. \quad (35)$$

On the other hand, for the BSC, the only symbol in the set \mathcal{X}_{2T} is symbol 1, which RV V takes with probability one. In other words, $p_V(x_2)$ is a degenerate distribution, given by $p_V(x_2) = \delta(x_2 - 1)$. Hence,

$$\max_{p_V(x_2)} I(X_2; Y_2) = I(X_2; Y_2) \Big|_{p_V(x_2)=\delta(x_2-1)} \quad (36)$$

$$= H(Y_2) \Big|_{p_V(x_2)=\delta(x_2-1)} - H(Y_2|X_2) \Big|_{p_V(x_2)=\delta(x_2-1)}. \quad (37)$$

For the BSC, the expression for $H(Y_2|X_2)$ is independent of X_2 , and is given by [1]

$$H(Y_2|X_2) = H(P_{\varepsilon 2}). \quad (38)$$

On the other hand, in order to find $H(Y_2) \Big|_{p_V(x_2)=\delta(x_2-1)}$ from (16), we need the distributions of $p(y_2|x_2 = 0)$ and $p(y_2|x_2 = 1)$. For the BSC with probability of error $P_{\varepsilon 2}$, these distributions are obtained as

$$p(y_2|x_2 = 0) = \begin{cases} 1 - P_{\varepsilon 2} & \text{if } y_2 = 0 \\ P_{\varepsilon 2} & \text{if } y_2 = 1, \end{cases} \quad (39)$$

and

$$p(y_2|x_2 = 1) = \begin{cases} P_{\varepsilon 2} & \text{if } y_2 = 0 \\ 1 - P_{\varepsilon 2} & \text{if } y_2 = 1. \end{cases} \quad (40)$$

Inserting (39), (40), and $p_V(x_2) = \delta(x_2 - 1)$ into (16), we obtain $H(Y_2) \Big|_{p_V(x_2)=\delta(x_2-1)}$ as

$$H(Y_2) \Big|_{p_V(x_2)=\delta(x_2-1)} = -A \log_2(A) - (1 - A) \log_2(1 - A), \quad (41)$$

where

$$A = P_{\varepsilon 2}(1 - 2P_U) + P_U. \quad (42)$$

Inserting (38) and (41) into (36), we obtain $\max_{p_V(x_2)} I(X_2; Y_2)$ as

$$\max_{p_V(x_2)} I(X_2; Y_2) = -A \log_2(A) - (1 - A) \log_2(1 - A) - H(P_{\varepsilon 2}). \quad (43)$$

We now have the two necessary components required for obtaining P_U^* from (18), and thereby obtaining the capacity. This is summarized in the following corollary.

Corollary 1: The capacity of the considered relay channel with BSCs links is given by

$$C = \max_{P_U} \min \left\{ (1 - H(P_{\varepsilon 1}))(1 - P_U), -A \log_2(A) - (1 - A) \log_2(1 - A) - H(P_{\varepsilon 2}) \right\} \quad (44)$$

and is achieved with

$$p_V(x_2) = \delta(x_2 - 1) \quad (45)$$

$$p(x_1 = 0 | x_2 = 0) = p(x_1 = 1 | x_2 = 0) = 1/2. \quad (46)$$

There are two cases for the optimal P_U^* which maximizes (44). If P_U found from⁵

$$(1 - H(P_{\varepsilon 1}))(1 - P_U) = -A \log_2(A) - (1 - A) \log_2(1 - A) - H(P_{\varepsilon 2}) \quad (47)$$

is smaller than 1/2, then the optimal P_U^* which maximizes (44) is found as the solution to (47), and the capacity simplifies to

$$C = (1 - H(P_{\varepsilon 1}))(1 - P_U^*) = -A^* \log_2(A^*) - (1 - A^*) \log_2(1 - A^*) - H(P_{\varepsilon 2}), \quad (48)$$

where $A^* = A|_{P_U=P_U^*}$. Otherwise, if P_U found from (47) is $P_U \geq 1/2$, then the optimal P_U^* which maximizes (44) is $P_U^* = 1/2$, and the capacity simplifies to

$$C = 1 - H(P_{\varepsilon 2}). \quad (49)$$

Proof: The capacity in (44) is obtained by inserting (33) and (43) into (18). On the other hand, for the BSC, the solution of (20) is $P_U'' = 1/2$, whereas (19) simplifies to (47). Hence, using Theorem 1, we obtain that if $P_U' \leq P_U'' = 1/2$, then $P_U^* = P_U'$, where P_U' is found from (47), in which case the capacity is given by (21), which simplifies to (48) for the BSC. On the other hand, if $P_U' > P_U'' = 1/2$, then $P_U^* = P_U'' = 1/2$, in which case the capacity is given by (22), which simplifies to (49) for the BSC. ■

⁵ Solving (47) with respect to P_U leads to a nonlinear equation, which can be easily solved using e.g. Newton's method [23].

B. AWGN Channels

In this subsection, we assume that the source-relay and relay-destination links are AWGN channels, i.e., channels which are impaired by independent, real-valued, zero-mean AWGN with variances σ_1^2 and σ_2^2 , respectively. More precisely, the outputs at the relay and the destination are given by

$$Y_k = X_k + N_k, \quad k \in \{1, 2\}, \quad (50)$$

where N_k is a zero-mean Gaussian RV with variance σ_k^2 , $k \in \{1, 2\}$, with distribution $p_{N_k}(z)$, $k \in \{1, 2\}$, $-\infty \leq z \leq \infty$. Moreover, assume that the symbols transmitted by the source and the relay must satisfy the following average power constraints⁶

$$\sum_{x_1 \in \mathcal{X}_1} x_1^2 p(x_1|x_2=0) \leq P_1 \quad \text{and} \quad \sum_{x_2 \in \mathcal{X}_{2T}} x_2^2 p_V(x_2) \leq P_2. \quad (51)$$

Obtaining the capacity for this relay channel using Theorem 1, requires expressions for the functions $\max_{p(x_1|x_2=0)} I(X_1; Y_1|X_2=0)$ and $\max_{p_V(x_2)} I(X_2; Y_2) = \max_{p_V(x_2)} [H(Y_2) - H(Y_2|X_2)]$. For the AWGN channel, the expressions for the mutual information $\max_{p(x_1|x_2=0)} I(X_1; Y_1|X_2=0)$ and the entropy $H(Y_2|X_2)$ are well known and given by

$$\max_{p(x_1|x_2=0)} I(X_1; Y_1|X_2=0) = \frac{1}{2} \log_2 \left(1 + \frac{P_1}{\sigma_1^2} \right) \quad (52)$$

$$H(Y_2|X_2) = \frac{1}{2} \log_2 (2\pi e \sigma_2^2), \quad (53)$$

where, as is well known, for AWGN $I(X_1; Y_1|X_2=0)$ is maximized when $p(x_1|x_2=0)$ is the zero mean Gaussian distribution with variance P_1 . On the other hand, $H(Y_2|X_2)$ is just the differential entropy of Gaussian RV N_2 , which is independent of $p(x_2)$, i.e., of $p_V(x_2)$. Hence,

$$\max_{p_V(x_2)} I(X_2; Y_2) = \max_{p_V(x_2)} H(Y_2) - \frac{1}{2} \log_2 (2\pi e \sigma_2^2) \quad (54)$$

holds and in order to find $\max_{p_V(x_2)} I(X_2; Y_2)$ we only need to derive $\max_{p_V(x_2)} H(Y_2)$. Now, in order to find $\max_{p_V(x_2)} H(Y_2)$, we first obtain $H(Y_2)$ using (16) and then obtain the distribution $p_V(x_2)$

⁶If the optimal distributions $p(x_1|x_2=0)$ and $p_V(x_2)$ turn out to be continuous, the sums in (51) should be replaced by integrals. We note however that the generalization of capacity expressions for a discrete to a continuous channel model may not be always straightforward. An exception to this is the AWGN channel which has been well studied in the literature [21].

which maximizes $H(Y_2)$. Finding an expression for $H(Y_2)$ requires the distribution of $p(y_2|x_2)$. This distribution is found using (50) as

$$p(y_2|x_2) = p_{N_2}(y_2 - x_2). \quad (55)$$

Inserting (55) into (16), we obtain $H(Y_2)$ as

$$\begin{aligned} H(Y_2) = & - \int_{-\infty}^{\infty} \left[P_U \sum_{x_2 \in \mathcal{X}_{2T}} p_{N_2}(y_2 - x_2) p_V(x_2) + p_{N_2}(y_2)(1 - P_U) \right] \\ & \times \log_2 \left[P_U \sum_{x_2 \in \mathcal{X}_{2T}} p_{N_2}(y_2 - x_2) p_V(x_2) + p_{N_2}(y_2)(1 - P_U) \right] dy_2, \end{aligned} \quad (56)$$

where, since $p(y_2|x_2)$ is now a continuous probability density function, the summation in (16) with respect to y_2 converges to an integral as

$$\sum_{y_2} \rightarrow \int_{-\infty}^{\infty} dy_2. \quad (57)$$

We are now ready to maximize $H(Y_2)$ in (56) with respect to $p_V(x_2)$. Unfortunately, obtaining the optimal $p_V(x_2)$ which maximizes $H(Y_2)$ in closed form is difficult, if not impossible. However, as will be shown in the following lemma, we still can characterize the optimal $p_V(x_2)$, which is helpful for numerical calculation of $p_V(x_2)$.

Lemma 1: For the considered relay channel where the relay-destination link is an AWGN channel and where the input symbols of the relay must satisfy the average power constraint given in (51), the distribution $p_V(x_2)$ which maximizes $H(Y_2)$ in (56) for a fixed $P_U < 1$ is discrete, symmetric around zero, and with infinite number of mass points, where the probability mass points in any bounded interval is finite, i.e., $p_V(x_2)$ has the following form

$$p_V(x_2) = \sum_{k=1}^{\infty} p_k \delta(x_2 - x_{2k}), \quad (58)$$

where p_k is the probability that symbol x_2 will take the value x_{2k} , for $k = 1, \dots, \infty$. Furthermore, p_k and x_{2k} given in (58), must satisfy

$$\sum_{k=1}^{\infty} p_k = 1 \quad \text{and} \quad \sum_{k=1}^{\infty} p_k x_{2k}^2 = P_2. \quad (59)$$

In the limiting case when $P_U \rightarrow 1$, distribution $p_V(x_2)$ converges to the zero-mean Gaussian distribution with variance P_2 .

Proof: Please see Appendix B. ■

Remark 1: Unfortunately, there is no closed-form expression for distribution $p_V(x_2)$ given in the form of (58), and therefore, a brute-force search has to be used in order to find x_{2k} and p_k , $\forall k$.

Now, inserting (58) into (56) we obtain $\max_{p_V(x_2)} H(Y_2)$ as

$$\begin{aligned} \max_{p_V(x_2)} H(Y_2) = & - \int_{-\infty}^{\infty} \left(P_U \sum_{k=1}^{\infty} p_k^* p_{N_2}(y_2 - x_{2k}^*) + (1 - P_U) p_{N_2}(y_2) \right) \\ & \times \log_2 \left(P_U \sum_{k=1}^{\infty} p_k^* p_{N_2}(y_2 - x_{2k}^*) + (1 - P_U) p_{N_2}(y_2) \right) dy_2, \quad (60) \end{aligned}$$

where $p_V^*(x_2) = \sum_{k=1}^{\infty} p_k^* \delta(x_2 - x_{2k}^*)$ is the distribution that maximizes $H(Y_2)$ in (56). Inserting (60) into (54), we obtain $\max_{p_V(x_2)} I(X_2; Y_2)$. Using (52) and $\max_{p_V(x_2)} I(X_2; Y_2)$ in Theorem 1, we obtain the capacity of the considered relay channel with AWGN links. This is conveyed in the following corollary.

Corollary 2: The capacity of the considered relay channel where the source-relay and relay-destination links are both AWGN channels with noise variances σ_1^2 and σ_2^2 , respectively, and where the average power constraints of the inputs of source and relay are given by (51), is given by

$$\begin{aligned} C &= \frac{1}{2} \log_2 \left(1 + \frac{P_1}{\sigma_1^2} \right) (1 - P_U^*) \\ &\stackrel{(a)}{=} - \int_{-\infty}^{\infty} \left(P_U^* \sum_{k=1}^{\infty} p_k^* p_{N_2}(y_2 - x_{2k}^*) + (1 - P_U^*) p_{N_2}(y_2) \right) \\ &\quad \times \log_2 \left(P_U^* \sum_{k=1}^{\infty} p_k^* p_{N_2}(y_2 - x_{2k}^*) + (1 - P_U^*) p_{N_2}(y_2) \right) dy_2 - \frac{1}{2} \log_2(2\pi e \sigma_2^2), \quad (61) \end{aligned}$$

where the optimal P_U^* is found such that equality (a) in (61) holds. The capacity in (61) is achieved when $p(x_1|x_2 = 0)$ is the zero-mean Gaussian distribution with variance P_1 and $p_V^*(x_2) = \sum_{k=1}^{\infty} p_k^* \delta(x_2 - x_{2k}^*)$ is a discrete distribution which satisfies (59) and maximizes $H(Y_2)$ given in (60).

Proof: The capacity in (61) is obtained by inserting (60) into (54), then inserting (54) and (52) into (18), and finally maximizing with respect to P_U . For the maximization of the corresponding capacity with respect to P_U , we note that $P_U' < P_U'' = 1$ always holds. Hence, the capacity is given by (21), which for the Gaussian case simplifies to (61). To see that $P_U'' =$

1, note the relay-destination channel is an AWGN channel for which the mutual information is maximized when $p(x_2)$ is a Gaussian distribution. From (8), we see that $p(x_2)$ becomes a Gaussian distribution if and only if $P_U = 1$ and $p_V(x_2)$ also assumes a Gaussian distribution. ■

V. NUMERICAL EXAMPLES

In this section, we numerically evaluate the capacities of the considered HD relay channel when the source-relay and relay-destination links are both BSCs and AWGN channels, respectively. As a performance benchmark, we use the maximal achievable rate of conventional relaying [3]. Thereby, the source transmits to the relay one codeword with rate $\max_{p(x_1|x_2=0)} I(X_1; Y_1|X_2 = 0)$ in $1 - P_U$ fraction of the time, where $0 < P_U < 1$, and in the remaining fraction of time, P_U , the relay retransmits the received information to the destination with rate $\max_{p(x_2)} I(X_2; Y_2)$, see [2] and [3]. The optimal P_U , is found such that the following holds

$$R_{\text{conv}} = \max_{p(x_1|x_2=0)} I(X_1; Y_1|X_2 = 0)(1 - P_U) = \max_{p(x_2)} I(X_2; Y_2)P_U. \quad (62)$$

Employing the optimal P_U obtained from (62), the maximal achievable rate of conventional relaying can be written as

$$R_{\text{conv}} = \frac{\max_{p(x_1|x_2=0)} I(X_1; Y_1|X_2 = 0) \times \max_{p(x_2)} I(X_2; Y_2)}{\max_{p(x_1|x_2=0)} I(X_1; Y_1|X_2 = 0) + \max_{p(x_2)} I(X_2; Y_2)}. \quad (63)$$

A. BSC Links

For simplicity, we assume symmetric links with $P_{\varepsilon 1} = P_{\varepsilon 2} = P_\varepsilon$. As a result, $P_U^* < 1/2$ in Corollary 1 and the capacity is given by (48). This capacity is plotted in Fig. 5, where P_U^* is found from (47) using a mathematical software package, e.g. Mathematica. As a benchmark, in Fig. 5, we also show the maximal achievable rate using conventional relaying, obtained by inserting

$$\max_{p(x_1|x_2=0)} I(X_1; Y_1|X_2 = 0) = \max_{p(x_2)} I(X_2; Y_2) = 1 - H(P_\varepsilon) \quad (64)$$

into (63), where $H(P_\varepsilon)$ is given in (34) with $P = P_\varepsilon$. Thereby, the following rate is obtained

$$R_{\text{conv}} = \frac{1}{2}(1 - H(P_\varepsilon)). \quad (65)$$

As can be seen from Fig. 5, when both links are error-free, i.e., $P_\varepsilon = 0$, conventional relaying achieves 0.5 bits/channel use, whereas the capacity is 0.77291, which is 54% larger than the rate

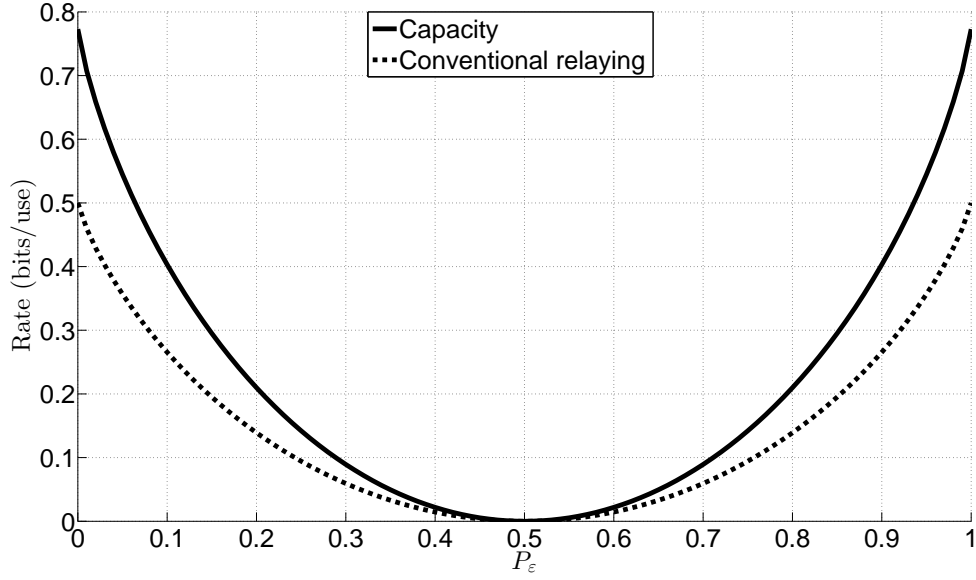


Fig. 5. Comparison of rates for the BSC as a function of the error probability $P_{\varepsilon 1} = P_{\varepsilon 2} = P_{\varepsilon}$.

achieved with conventional relaying. This value for the capacity can be obtained by inserting $P_{\varepsilon 1} = P_{\varepsilon 2} = 0$ in (47), and thereby obtain

$$C = 1 - P_U^* \stackrel{(a)}{=} H(P_U^*). \quad (66)$$

Solving (a) in (66) with respect to P_U^* and inserting the solution for P_U^* back into (66), yields $C = 0.77291$. We note that this value was first reported in [6, page 327].

B. AWGN Links

For the AWGN case, the capacity is evaluated based on Corollary 2. However, since for this case the optimal input distribution at the relay $p_V^*(x_2)$ is unknown, i.e., the values of p_k^* and x_{2k}^* in (61) are unknown, we have performed a brute force search for the values of p_k^* and x_{2k}^* which maximize (61). Two examples of such distributions⁷ are shown in Fig. 6 for two different values of the SNR $P_1/\sigma_1^2 = P_2/\sigma_2^2$. Since we do not have a proof that the distributions obtained via brute-force search are actually the exact optimal input distributions at the relay that achieve the capacity, the rates that we obtain, denoted by C_L , are lower than or equal to the actual

⁷ Note that these distributions resemble a discrete, Gaussian shaped distribution with a gap around zero.

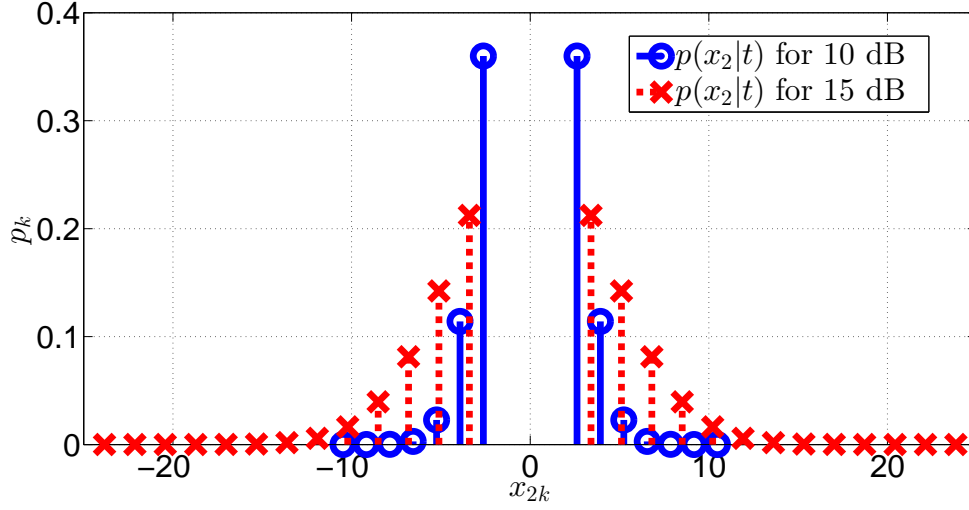


Fig. 6. Example of proposed input distributions at the relay $p_V(x_2)$.

capacity. These rates are shown in Figs. 7 and 8, for symmetric links and non-symmetric links, respectively, where we set $P_1/\sigma_1^2 = P_2/\sigma_2^2$ and $P_1/\sigma_1^2/10 = P_2/\sigma_2^2$, respectively. We note that for the results in Fig. 7, for $P_1/\sigma_1^2 = P_2/\sigma_2^2 = 10$ dB and $P_1/\sigma_1^2 = P_2/\sigma_2^2 = 15$ dB, we have used the input distributions at the relay shown in Fig. 6. In particular, for $P_1/\sigma_1^2 = P_2/\sigma_2^2 = 10$ dB we have used the following values for p_k^* and x_{2k}^*

$$p_k^* = [0.35996, 0.11408, 2.2832 \times 10^{-2}, 2.88578 \times 10^{-3}, 2.30336 \times 10^{-4}, \\ 1.16103 \times 10^{-5}, 3.69578 \times 10^{-7}],$$

$$x_{2k}^* = [2.62031, 3.93046, 5.24061, 6.55077, 7.86092, 9.17107, 10.4812],$$

and for $P_1/\sigma_1^2 = P_2/\sigma_2^2 = 15$ dB we have used

$$p_k^* = [0.212303, 0.142311, 8.12894 \times 10^{-2}, 3.95678 \times 10^{-2}, 1.64121 \times 10^{-2}, 5.80092 \times 10^{-3}, \\ 1.7472 \times 10^{-3}, 4.48438 \times 10^{-4}, 9.80788 \times 10^{-5}, 1.82793 \times 10^{-5}, 2.90308 \times 10^{-6}, 3.92889 \times 10^{-7}], \\ x_{2k}^* = [3.40482, 5.10724, 6.80965, 8.51206, 10.2145, 11.9169, 13.6193, 15.3217, \\ 17.0241, 18.7265, 20.4289, 22.1314].$$

The above values of p_k^* and x_{2k}^* are only given for $x_{2k}^* > 0$, since the values of p_k^* and x_{2k}^* when $x_{2k}^* < 0$ can be found from symmetry, see Fig. 6.

In Figs. 7 and 8, we also show the rate achieved when instead of an optimal discrete input distribution at the relay $p_V^*(x_2)$, cf. Lemma 1, we use a continuous, zero-mean Gaussian

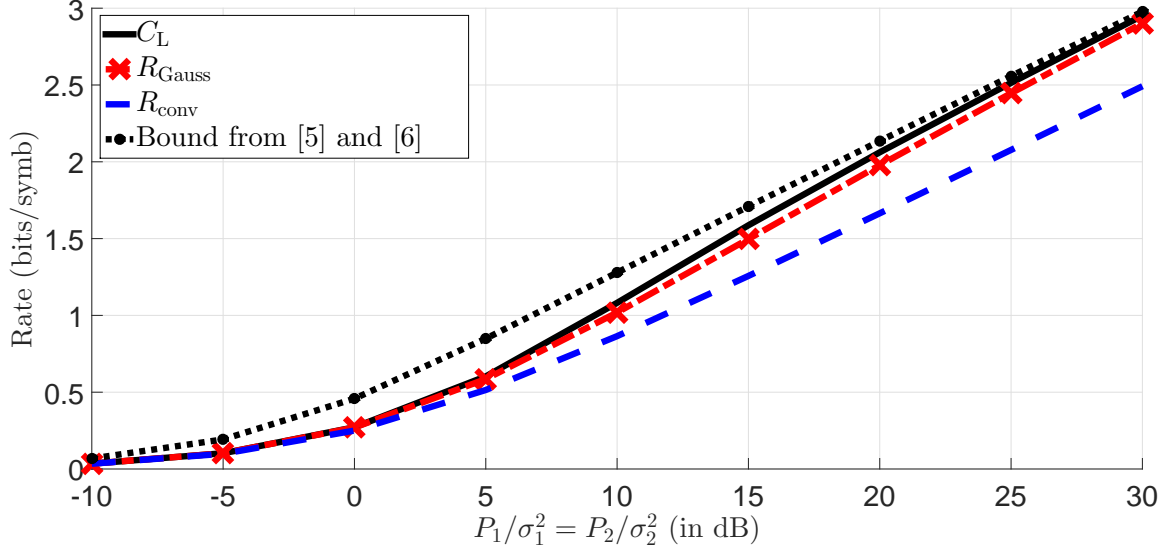


Fig. 7. Source-relay and relay destination links are AWGN channels with $P_1/\sigma_1^2 = P_2/\sigma_2^2$.

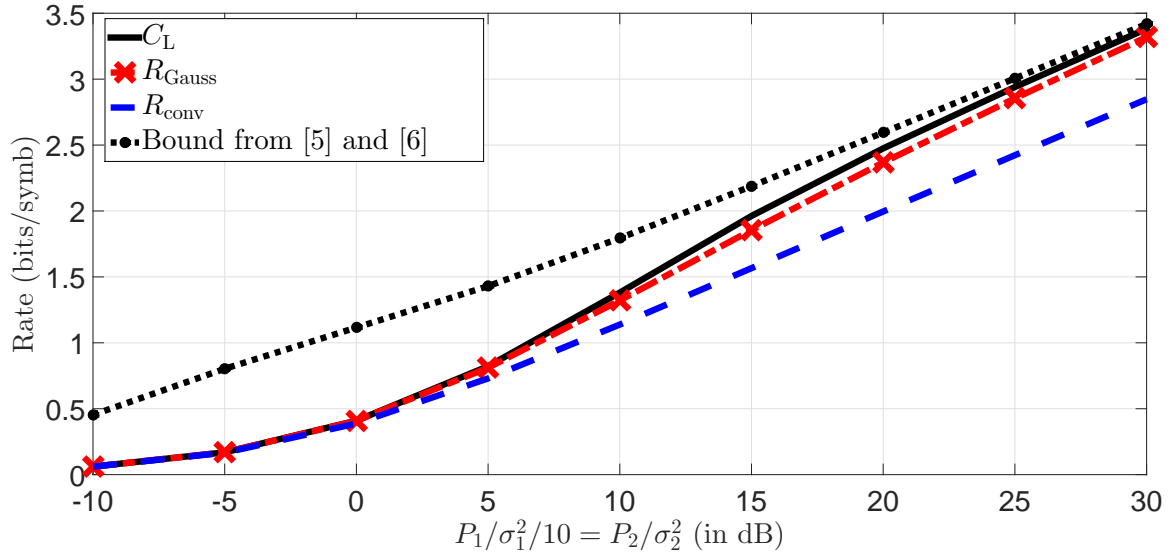


Fig. 8. Source-relay and relay destination links are AWGN channels with $P_1/\sigma_1^2/10 = P_2/\sigma_2^2$.

distribution with variance P_2 . Thereby, we obtain the following rate

$$\begin{aligned}
 R_{\text{Gauss}} &= \frac{1}{2} \log_2 \left(1 + \frac{P_1}{\sigma_1^2} \right) (1 - P_U) \\
 &\stackrel{(a)}{=} - \int_{-\infty}^{\infty} (P_U p_G(y_2) + (1 - P_U) p_{N_2}(y_2)) \\
 &\quad \times \log_2 (P_U p_G(y_2) + (1 - P_U) p_{N_2}(y_2)) dy_2 - \frac{1}{2} \log_2 (2\pi e \sigma_2^2), \quad (67)
 \end{aligned}$$

where P_U is found such that equality (a) holds and $p_G(y_2)$ is a continuous, zero-mean Gaussian distribution with variance $P_2 + \sigma_2^2$. From Figs. 7 and 8, we can see that $R_{\text{Gauss}} \leq C_L$, which was expected from Lemma 1. However, the loss in performance caused by the Gaussian inputs is moderate, which suggests that the performance gains obtained by the proposed protocol are mainly due to the exploitation of the silent (zero) symbols for conveying information from the HD relay to the destination rather than the optimization of $p_V(x_2)$.

As benchmark, in Figs. 7 and 8, we have also shown the maximal achievable rate using conventional relaying, obtained by inserting

$$\max_{p(x_1|x_2=0)} I(X_1; Y_1 | X_2 = 0, U = r) = \frac{1}{2} \log_2 \left(1 + \frac{P_1}{\sigma_1^2} \right) \quad (68)$$

and

$$\max_{p_V(x_2)} I(X_2; Y_2 | U = t) = \frac{1}{2} \log_2 \left(1 + \frac{P_2}{\sigma_2^2} \right) \quad (69)$$

into (63), which yields

$$R_{\text{conv}} = \frac{1}{2} \frac{\log_2 \left(1 + \frac{P_1}{\sigma_1^2} \right) \log_2 \left(1 + \frac{P_2}{\sigma_2^2} \right)}{\log_2 \left(1 + \frac{P_1}{\sigma_1^2} \right) + \log_2 \left(1 + \frac{P_2}{\sigma_2^2} \right)}. \quad (70)$$

Comparing the rates C_L and R_{conv} in Figs. 7 and 8, we see that for $10 \text{ dB} \leq P_2/\sigma_2^2 \leq 30 \text{ dB}$, C_L achieves 3 to 6 dB gain compared to R_{conv} . Hence, large performance gains are achieved using the proposed capacity protocol even if suboptimal input distributions at the relay are employed.

Finally, as additional benchmark in Figs. 7 and 8, we show the upper bounds, achievable to withing 1 bit/symbol, reported in [5] and [6], given by

$$C_{\text{Upper}} = \max_{P_U} \min \left\{ \frac{1}{2} \log_2 \left(1 + \frac{P_1}{\sigma_1^2} \right) (1 - P_U), \frac{1}{2} \log_2 \left(1 + \frac{P_2}{\sigma_2^2} \right) P_U + H(P_U) \right\}. \quad (71)$$

As can be seen from Figs. 7 and 8, this bound is loose for low SNRs but becomes tight for high SNRs.

VI. CONCLUSION

We have derived an easy-to-evaluate expression for the capacity of the two-hop HD relay channel by simplifying previously derived expression for the converse. Moreover, we have proposed an explicit coding scheme which achieves the capacity. In particular, we showed that the capacity is achieved when the relay sends additional information to the destination by using

the zero symbol implicitly generated by the relay's silence during reception. Furthermore, we have evaluated the capacity for the cases when both links are BSCs and AWGN channels, respectively. From the numerical examples, we have observed that the capacity of the two-hop HD relay channel is significantly higher than the rates achieved with conventional relaying protocols.

APPENDIX

A. Proof That the Probability of Error at the Relay Goes to Zero When (31) Holds

In order to prove that the relay can decode the source's codeword in block b , $\mathbf{x}_{1|r}(b)$, where $1 \leq b \leq N$, from the received codeword $\mathbf{y}_{1|r}(b)$ when (31) holds, i.e., that the probability of error at the relay goes to zero as $k \rightarrow \infty$, we will follow the "standard" method in [21, Sec. 7.7] for analyzing the probability of error for rates smaller than the capacity. To this end, note that the length of codeword $\mathbf{x}_{1|r}(b)$ is $k(1 - P_U^*)$. On the other hand, the length of codeword $\mathbf{y}_{1|r}(b)$ is identical to the number of zeros⁸ in relay's transmit codeword $\mathbf{x}_2(b)$. Since the zeros in $\mathbf{x}_2(b)$ are generated independently using a coin flip, the number of zeros, i.e., the length of $\mathbf{y}_{1|r}(b)$ is $k(1 - P_U^*) \pm \varepsilon(b)$, where $\varepsilon(b)$ is a non-negative integer. Due to the strong law of large numbers, the following holds

$$\lim_{k \rightarrow \infty} \frac{\varepsilon(b)}{k} = 0, \quad (72)$$

$$\lim_{k \rightarrow \infty} \frac{k(1 - P_U^*) \pm \varepsilon(b)}{k(1 - P_U^*)} = 1, \quad (73)$$

i.e., for large k , the length of relay's received codeword $\mathbf{y}_{1|r}(b)$ is approximately $k(1 - P_U^*)$.

Now, for block b , we define a set $\mathcal{R}(b)$ which contains the symbol indices i in block b for which the symbols in $\mathbf{x}_2(b)$ are zeros, i.e., for which $X_{2i} = 0$. Note that before the start of the transmission in block b , the relay knows $\mathbf{x}_2(b)$, thereby it knows a priori for which symbol indices i in block b , $X_{2i} = 0$ holds. Furthermore, note that

$$|\mathcal{R}(b)| = k(1 - P_U^*) \pm \varepsilon(b) \quad (74)$$

holds, where $|\cdot|$ denotes cardinality of a set. Depending on the relation between $|\mathcal{R}(b)|$ and $k(1 - P_U^*)$, the relay has to distinguish between two cases for decoding $\mathbf{x}_{1|r}(b)$ from $\mathbf{y}_{1|r}(b)$.

⁸For $b = 1$, note that the number of zeros in $\mathbf{x}_2(1)$ is k . Therefore, for $b = 1$, we only take into an account the first $k(1 - P_U^*)$ zeros. As a result, the length of $\mathbf{y}_{1|r}(b)$ is also $k(1 - P_U^*)$.

In the first case $|\mathcal{R}(b)| \geq k(1 - P_U^*)$ holds whereas in the second case $|\mathcal{R}(b)| < k(1 - P_U^*)$ holds. We first explain the decoding procedure for the first case.

When $|\mathcal{R}(b)| = k(1 - P_U^*) + \varepsilon(b) \geq k(1 - P_U^*)$ holds, the source can transmit the entire codeword $\mathbf{x}_{1|r}(b)$, which is comprised of $k(1 - P_U^*)$ symbols, since there are enough zeros in codeword $\mathbf{x}_2(b)$. On the other hand, since for this case the received codeword $\mathbf{y}_{1|r}(b)$ is comprised of $k(1 - P_U^*) + \varepsilon(b)$ symbols, and since for the last $\varepsilon(b)$ symbols in $\mathbf{y}_{1|r}(b)$ the source is silent, the relay keeps from $\mathbf{y}_{1|r}(b)$ only the first $k(1 - P_U^*)$ symbols and discards the remanining $\varepsilon(b)$ symbols. In this way, the relay keeps only the received symbols which are the result of the transmitted symbols in $\mathbf{x}_{1|r}(b)$, and discards the rest of the symbols in $\mathbf{y}_{1|r}(b)$ for which the source is silent. Thereby, from $\mathbf{y}_{1|r}(b)$, the relay generates a new received codeword which we denote by $\mathbf{y}_{1|r}^*(b)$. Moreover, let $\mathcal{R}_1(b)$ be a set which contains the symbol indices of the symbols comprising codeword $\mathbf{y}_{1|r}^*(b)$. Now, note that the lengths of $\mathbf{x}_{1|r}(b)$ and $\mathbf{y}_{1|r}^*(b)$, and the cardinality of set $\mathcal{R}_1(b)$ are $k(1 - P_U^*)$, respectively. Having created $\mathbf{y}_{1|r}^*(b)$ and $\mathcal{R}_1(b)$, we now use a jointly typical decoder for decoding $\mathbf{x}_{1|r}(b)$ from $\mathbf{y}_{1|r}^*(b)$. In particular, we define a jointly typical set $A_\epsilon^{|\mathcal{R}_1(b)|}$ as

$$A_\epsilon^{|\mathcal{R}_1(b)|} = \left\{ (\mathbf{x}_{1|r}, \mathbf{y}_{1|r}^*) \in \mathcal{X}_1^{|\mathcal{R}_1(b)|} \times \mathcal{Y}_1^{|\mathcal{R}_1(b)|} : \right. \\ \left. \left| -\frac{1}{|\mathcal{R}_1(b)|} \sum_{i \in \mathcal{R}_1(b)} \log_2 p(x_{1i} | x_{2i} = 0) - H(X_1 | X_2 = 0) \right| \leq \epsilon, \right. \quad (75a)$$

$$\left. \left| -\frac{1}{|\mathcal{R}_1(b)|} \sum_{i \in \mathcal{R}_1(b)} \log_2 p(y_{1i} | x_{2i} = 0) - H(Y_1 | X_2 = 0) \right| \leq \epsilon, \right. \quad (75b)$$

$$\left. \left| -\frac{1}{|\mathcal{R}_1(b)|} \sum_{i \in \mathcal{R}_1(b)} \log_2 p(x_{1i}, y_{1i} | x_{2i} = 0) - H(X_1, Y_1 | X_2 = 0) \right| \leq \epsilon \right\}, \quad (75c)$$

where ϵ is a small positive number. The transmitted codeword $\mathbf{x}_{1|r}(b)$ is successfully decoded from received codeword $\mathbf{y}_{1|r}^*(b)$ if and only if $(\mathbf{x}_{1|r}(b), \mathbf{y}_{1|r}^*(b)) \in A_\epsilon^{|\mathcal{R}_1(b)|}$ and no other codeword $\hat{\mathbf{x}}_{1|r}$ from codebook $\mathcal{C}_{1|r}$ is jointly typical with $\mathbf{y}_{1|r}^*(b)$. In order to compute the probability of error, we define the following events

$$E_0 = \{(\mathbf{x}_{1|r}(b), \mathbf{y}_{1|r}^*(b)) \notin A_\epsilon^{|\mathcal{R}_1(b)|}\} \text{ and } E_j = \{(\hat{\mathbf{x}}_{1|r}^{(j)}, \mathbf{y}_{1|r}^*(b)) \in A_\epsilon^{|\mathcal{R}_1(b)|}\}, \quad (76)$$

where $\hat{\mathbf{x}}_{1|r}^{(j)}$ is the j -th codeword in $\mathcal{C}_{1|r}$ that is different from $\mathbf{x}_{1|r}(b)$. Note that in $\mathcal{C}_{1|r}$ there are $|\mathcal{C}_{1|r}| - 1 = 2^{kR} - 1$ codewords that are different from $\mathbf{x}_{1|r}(b)$, i.e., $j = 1, \dots, 2^{kR} - 1$. Hence, an

error occurs if any of the events $E_0, E_1, \dots, E_{2^{kR}-1}$ occurs. Since $\mathbf{x}_{1|r}(b)$ is uniformly selected from the codebook $\mathcal{C}_{1|r}$, the average probability of error is given by

$$\Pr(\epsilon) = \Pr(E_0 \cup E_1 \cup \dots \cup E_{2^{kR}-1}) \leq \Pr(E_0) + \sum_{j=1}^{2^{kR}-1} \Pr(E_j). \quad (77)$$

Since $|\mathcal{R}_1(b)| \rightarrow \infty$ as $k \rightarrow \infty$, $\Pr(E_0)$ in (77) is upper bounded as [21, Eq. (7.74)]

$$\Pr(E_0) \leq \epsilon. \quad (78)$$

On the other hand, since $|\mathcal{R}_1(b)| \rightarrow \infty$ as $k \rightarrow \infty$, $\Pr(E_j)$ is upper bounded as

$$\begin{aligned} \Pr(E_j) &= \Pr\left(\left(\hat{\mathbf{x}}_{1|r}^{(j)}, \mathbf{y}_{1|r}^*(b)\right) \in A_\epsilon^{|\mathcal{R}_1(b)|}\right) = \sum_{(\hat{\mathbf{x}}_{1|r}^{(j)}, \mathbf{y}_{1|r}^*(b)) \in A_\epsilon^{|\mathcal{R}_1(b)|}} p(\hat{\mathbf{x}}_{1|r}^{(j)}, \mathbf{y}_{1|r}^*(b)) \\ &\stackrel{(a)}{=} \sum_{(\hat{\mathbf{x}}_{1|r}^{(j)}, \mathbf{y}_{1|r}^*(b)) \in A_\epsilon^{|\mathcal{R}_1(b)|}} p(\hat{\mathbf{x}}_{1|r}^{(j)}) p(\mathbf{y}_{1|r}^*(b)) \\ &\stackrel{(b)}{\leq} \sum_{(\hat{\mathbf{x}}_{1|r}^{(j)}, \mathbf{y}_{1|r}^*(b)) \in A_\epsilon^{|\mathcal{R}_1(b)|}} 2^{-|\mathcal{R}_1(b)|(H(X_1|X_2=0)-\epsilon)} 2^{-|\mathcal{R}_1(b)|(H(Y_1|X_2=0)-\epsilon)} \\ &= |A_\epsilon^{|\mathcal{R}_1(b)|}| 2^{-|\mathcal{R}_1(b)|(H(X_1|X_2=0)-\epsilon)} 2^{-|\mathcal{R}_1(b)|(H(Y_1|X_2=0)-\epsilon)} \\ &\stackrel{(c)}{\leq} 2^{|\mathcal{R}_1(b)|(H(X_1, Y_1|X_2=0)+\epsilon)} 2^{-|\mathcal{R}_1(b)|(H(X_1|X_2=0)-\epsilon)} 2^{-|\mathcal{R}_1(b)|(H(Y_1|X_2=0)-\epsilon)} \\ &= 2^{-|\mathcal{R}_1(b)|(H(X_1|X_2=0)+H(Y_1|X_2=0)-H(X_1, Y_1|X_2=0)-3\epsilon)} \\ &= 2^{-|\mathcal{R}_1(b)|(I(X_1; Y_1|X_2=0)-3\epsilon)}, \end{aligned} \quad (79)$$

where (a) follows since $\hat{\mathbf{x}}_{1|r}^{(j)}$ and $\mathbf{y}_{1|r}^*(b)$ are independent, (b) follows since

$$p(\hat{\mathbf{x}}_{1|r}^{(j)}) \leq 2^{-|\mathcal{R}_1(b)|(H(X_1|X_2=0)-\epsilon)} \quad \text{and} \quad p(\mathbf{y}_{1|r}^*(b)) \leq 2^{-|\mathcal{R}_1(b)|(H(Y_1|X_2=0)-\epsilon)},$$

which follows from [21, Eq. (3.6)], respectively, and (c) follows since

$$|A_\epsilon^{|\mathcal{R}_1(b)|}| \leq 2^{|\mathcal{R}_1(b)|(H(X_1, Y_1|X_2=0)+\epsilon)},$$

which follows from [21, Theorem 7.6.1]. Inserting (78) and (79) into (77), we obtain

$$\begin{aligned} \Pr(\epsilon) &\leq \epsilon + \sum_{j=1}^{2^{kR}-1} 2^{-|\mathcal{R}_1(b)|(I(X_1; Y_1|X_2=0)-3\epsilon)} \\ &\leq \epsilon + (2^{kR} - 1) 2^{-|\mathcal{R}_1(b)|(I(X_1; Y_1|X_2=0)-3\epsilon)} \\ &\leq \epsilon + 2^{kR} 2^{-|\mathcal{R}_1(b)|(I(X_1; Y_1|X_2=0)-3\epsilon)} \\ &= \epsilon + 2^{-k((|\mathcal{R}_1(b)|/k)I(X_1; Y_1|X_2=0)-R-3(1-P_U^*)\epsilon)}. \end{aligned} \quad (80)$$

Hence, if

$$\begin{aligned} R &< \frac{|\mathcal{R}(b)|}{k} I(X_1; Y_1 | X_2 = 0) - 3(1 - P_U^*)\epsilon \\ &= (1 - P_U^*) I(X_1; Y_1 | X_2 = 0) - 3(1 - P_U^*)\epsilon, \end{aligned} \quad (81)$$

then $\lim_{\epsilon \rightarrow 0} \lim_{k \rightarrow \infty} \Pr(\epsilon) = 0$. This concludes the proof for case when $|\mathcal{R}(b)| \geq k(1 - P_U^*)$ holds. We now turn to case two when $|\mathcal{R}(b)| < k(1 - P_U^*)$ holds.

When $|\mathcal{R}(b)| = k(1 - P_U^*) - \varepsilon(b) < k(1 - P_U^*)$ holds, then the source cannot transmit all of its $k(1 - P_U^*)$ symbols comprising codeword $\mathbf{x}_{1|r}(b)$ since there are not enough zeros in codeword $\mathbf{x}_2(b)$. Instead, the relay transmits only $k(1 - P_U^*) - \varepsilon(b)$ symbols of codeword $\mathbf{x}_{1|r}(b)$, and we denote the resulting transmitted codeword by $\mathbf{x}_{1|r}^*(b)$. Note that the length of codewords $\mathbf{x}_{1|r}^*(b)$ and $\mathbf{y}_{1|r}(b)$, and the cardinality of $\mathcal{R}(b)$ are all identical and equal to $k(1 - P_U^*) - \varepsilon(b)$. In addition, let the relay generate a codebook $\mathcal{C}_{1|r}^*(b)$ by keeping only the first $k(1 - P_U^*) - \varepsilon(b)$ symbols from each codeword in codebook $\mathcal{C}_{1|r}$ and discarding the remaining $\varepsilon(b)$ symbols in the corresponding codewords. Let us denote the codewords in $\mathcal{C}_{1|r}^*(b)$ by $\mathbf{x}_{1|r}^*$. Note that there is a unique one to one mapping from the codewords in $\mathcal{C}_{1|r}^*(b)$ to the codewords in $\mathcal{C}_{1|r}(b)$ since when $k \rightarrow \infty$, $k(1 - P_U^*) - \varepsilon(b) \rightarrow \infty$ also holds, i.e., the lengths of the codewords in $\mathcal{C}_{1|r}^*(b)$ and $\mathcal{C}_{1|r}$ are of the same order due to (73). Hence, if the relay can decode $\mathbf{x}_{1|r}^*(b)$ from $\mathbf{y}_{1|r}(b)$, then using this unique mapping between $\mathcal{C}_{1|r}^*(b)$ and $\mathcal{C}_{1|r}(b)$, the relay can decode $\mathbf{x}_{1|r}(b)$ and thereby decode the message $w(b)$ sent from the source.

Now, for decoding $\mathbf{x}_{1|r}^*(b)$ from $\mathbf{y}_{1|r}(b)$, we again use jointly typical decoding. Thereby, we define a jointly typical set $B_\epsilon^{|\mathcal{R}|}$ as

$$B_\epsilon^{|\mathcal{R}|} = \left\{ (\mathbf{x}_{1|r}^*, \mathbf{y}_{1|r}) \in \mathcal{X}_1^{|\mathcal{R}|} \times \mathcal{Y}_1^{|\mathcal{R}|} : \right. \\ \left| -\frac{1}{|\mathcal{R}|} \sum_{i \in \mathcal{R}} \log_2 p(x_{1i} | x_{2i} = 0) - H(X_1 | X_2 = 0) \right| \leq \epsilon, \quad (82a)$$

$$\left| -\frac{1}{|\mathcal{R}|} \sum_{i \in \mathcal{R}} \log_2 p(y_{1i} | x_{2i} = 0) - H(Y_1 | X_2 = 0) \right| \leq \epsilon, \quad (82b)$$

$$\left. \left| -\frac{1}{|\mathcal{R}|} \sum_{i \in \mathcal{R}} \log_2 p(x_{1i}, y_{1i} | x_{2i} = 0) - H(X_1, Y_1 | X_2 = 0) \right| \leq \epsilon \right\}. \quad (82c)$$

Again, the transmitted codeword $\mathbf{x}_{1|r}^*(b)$ is successfully decoded from received codeword $\mathbf{y}_{1|r}(b)$ if and only if $(\mathbf{x}_{1|r}^*(b), \mathbf{y}_{1|r}(b)) \in B_\epsilon^{|\mathcal{R}|}$ and no other codeword $\hat{\mathbf{x}}_{1|r}^*$ from codebook $\mathcal{C}_{1|r}^*$ is

jointly typical with $\mathbf{y}_{1|r}(b)$. In order to compute the probability of error, we define the following events

$$E_0 = \{(\mathbf{x}_{1|r}^*(b), \mathbf{y}_{1|r}(b)) \notin B_\epsilon^{|\mathcal{R}|}\} \text{ and } E_j = \{(\hat{\mathbf{x}}_{1|r}^{*(j)}, \mathbf{y}_{1|r}(b)) \in B_\epsilon^{|\mathcal{R}|}\}, \quad (83)$$

where $\hat{\mathbf{x}}_{1|r}^{*(j)}$ is the j -th codeword in $\mathcal{C}_{1|r}^*$ that is different from $\mathbf{x}_{1|r}^*(b)$. Note that in $\mathcal{C}_{1|r}^*$ there are $|\mathcal{C}_{1|r}^*| - 1 = 2^{kR} - 1$ codewords that are different from $\mathbf{x}_{1|r}^*(b)$, i.e., $j = 1, \dots, 2^{kR} - 1$. Hence, an error occurs if any of the events $E_0, E_1, \dots, E_{2^{kR}-1}$ occurs. Now, using a similar procedure as for case when $|\mathcal{R}(b)| \geq k(1 - P_U^*)$, it can be proved that if

$$\begin{aligned} R &< \frac{|\mathcal{R}(b)|}{k} I(X_1; Y_1 | X_2 = 0) - 3(1 - P_U^*)\epsilon \\ &= (1 - P_U^*) I(X_1; Y_1 | X_2 = 0) - \frac{\epsilon(b)}{k} I(X_1; Y_1 | X_2 = 0) - 3(1 - P_U^*)\epsilon, \end{aligned} \quad (84)$$

then $\lim_{\epsilon \rightarrow 0} \lim_{k \rightarrow \infty} \Pr(\epsilon) = 0$. In (84), note that

$$\lim_{k \rightarrow \infty} \frac{\epsilon(b)}{k} I(X_1; Y_1 | X_2 = 0) = 0 \quad (85)$$

holds due to (72). This concludes the proof for the case when $|\mathcal{R}(b)| < k(1 - P_U^*)$.

B. Proof of Lemma 1

Lemma 1 is proven using the results from [24], where the authors investigate the optimal input distribution that achieves the capacity of an AWGN channel with an average power constraint P_2 and duty cycle q , where $0 < q < 1$. A duty cycle q means that from $n \rightarrow \infty$ symbol transmissions, at least q symbols have to be zero, see [24]. In other words, for the channel $Y_2 = X_2 + N_2$, the authors of [24] solve the following optimization problem

$$\begin{aligned} &\text{Maximize : } I(X_2; Y_2) = h(Y_2) - \frac{1}{2} \log_2(2\pi e \sigma_2^2) \\ &\quad \text{Subject to C1 : } E\{X_2^2\} \leq P_2, \\ &\quad \text{C2 : } \Pr\{X_2 = 0\} \geq q. \end{aligned} \quad (86)$$

Now, since $\frac{1}{2} \log_2(2\pi e \sigma_2^2)$ is independent of $p(x_2)$, $\Pr\{X_2 = 0\} \geq q$ is equivalent to $\Pr\{X_2 \neq 0\} = P_U \leq 1 - q$, and $p(x_2)$ is given by (8), the optimization problem in (86) can be written equivalently as

$$\begin{aligned} &\text{Maximize : } h(Y_2) \\ &\quad \text{Subject to C1 : } E\{X_2^2\} \leq P_2, \\ &\quad \text{C2 : } P_U \leq 1 - q. \end{aligned} \quad (87)$$

The authors in [24] prove that solving (87) for $q > 0$ yields a discrete distribution for $p_V(x_2)$, symmetric around zero, and with infinite number of mass points, where the probability mass points in any bounded interval is finite, see Theorem 2 in [24].

On the other hand, the optimization problem that we need to solve in order to prove Lemma 1 is

$$\begin{aligned} & \underset{p_V(x_2), P_U}{\text{Maximize :}} && h(Y_2) \\ & \text{Subject to} && \text{C1 : } E\{V_2^2\} \leq P_2, \\ & && \text{C2 : } P_U = 1 - q. \end{aligned} \tag{88}$$

Since $p(x_2)$ is given by (8), $E\{V_2^2\} = E\{X_2^2\}$ holds. As a result, optimization problem (88) can be written equivalently as

$$\begin{aligned} & \underset{p_V(x_2), P_U}{\text{Maximize :}} && h(Y_2) \\ & \text{Subject to} && \text{C1 : } E\{X_2^2\} \leq P_2, \\ & && \text{C2 : } P_U = 1 - q. \end{aligned} \tag{89}$$

Now, since we can always increase q in (87) such that constraint C2 in (87) holds with equality and since in that case again the optimal $p_V(x_2)$ of (87) is discrete, symmetric around zero, and with infinite number of mass points, where the probability mass points in any bounded interval is finite, we obtain that the optimal $p_V(x_2)$ of (89) also has to be discrete, symmetric around zero, and with infinite number of mass points, where the probability mass points in any bounded interval is finite. This concludes the proof of Lemma 1.

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